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STATIONARY CURRENTS IN OSCILLATING FLOWS IN TUBES IN THE CASE OF QUASISTATIONARY TURBULENCE

E. I. Permyakov

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It is well known that stationary flows are generated during the excitation of standing waves in resonators [1], substantially affecting heat and mass transfer [2, 3]. In the literature one can find a set of specific results in this field of study, but their range of application is comparatively narrow.

We clarify what we have in mind. Consider flows in resonating tubes. In the case of oscillating flows the current is characterized by two criteria: one usually uses the Strouhal number $Sh = 2R\omega/u_m$ and oscillation Reynolds number $Re_c = 2Ru_m/\nu$ (R is the tube radius, ω is the cyclic oscillation frequency, u_m is the velocity oscillation amplitude, and ν is the kinematic viscosity). In the Re_c–Sh plane one can indicate three regions: I — the laminar flow regions (in which are located all results available in the literature), II — the region of turbulent flows, in which the nonstationary character of turbulence is substantial, and III — the region in which turbulence can be assumed to be quasistationary (Fig. 1). The boundaries of these regions are curves 1 and 2, respectively, for

$$Sh = Re_c/160\,000; \tag{1}$$

$$Sh = 0{,}158/Re_c^{0{,}25} \tag{2}$$

The boundary (1) was obtained as a result of generalizing the experimental and theoretical data of [4], and that of (2) is the result of theoretical analysis [5]. The boundary (2) corresponds to the condition $Z = 4R\omega/\lambda u_m < 1$, where λ is the hydraulic resistance coefficient, and the dependence $\lambda(Re_c)$ in (2) is taken from the Blasius law for smooth-walled tubes [6]. Obviously, the broadening of the investigated region of secondary flows requires taking into account the possible flow turbulization.

In the present study we investigate theoretically stationary flows in the case of quasistationary turbulence, i.e., in region III, which can be extended substantially if the tube walls are rough, i.e., if λ = const [6].

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At one end of a long cylindrical tube ($R/L = \varepsilon \ll 1$) filled by a gas let there be located a source of harmonic oscillations (such as a planar piston), and let the other end be either closed or communicated with the surrounding medium. Let the amplitude of the resulting oscillations and the frequency be such that $u_m/\omega L \ll 1$, while Sh and Re_c are located in region III. Both these conditions can be satisfied simultaneously, since the first of them can be written in the form $\varepsilon \ll Sh/2$. We introduce the parameter $\Sigma = l_p/L = 2c_0^2/(\varkappa + 1)\omega L u_m$ – the ratio of the length of disruption formation to the length of the tube ($\varkappa$ is the adiabatic index, and c_0 is the speed of sound in the unperturbed gas). To solve the gas oscillation problem one can then use perturbation theory series for $\pi\sqrt{2\Sigma}|\omega - \omega_0|L/c_0 \geqslant 4$ (ω_0 is close to the resonance frequency ω) in the case of a closed tube [7] and when $\Sigma \gg 1$ if the tube is open at one end [8]. Neglecting nonlinear terms in the continuity, motion, and adiabat equations, we write down

$$\frac{\partial \bar{p}}{\partial y} = 0, \quad \bar{p} = \bar{\rho}, \quad \frac{R\omega}{c_0}\frac{\partial \bar{u}}{\partial \tau} = -\varepsilon\frac{\partial \bar{p}}{\partial z} + \frac{1}{c_0 R}\frac{1}{y}\frac{\partial}{\partial y}\left(\nu_{\text{eff}}\, y\frac{\partial \bar{u}}{\partial y}\right),$$
$$\frac{R\omega}{c_0}\frac{\partial \bar{\rho}}{\partial \tau} + \frac{1}{y}\frac{\partial}{\partial y}(y\bar{v}) + \varepsilon\frac{\partial \bar{u}}{\partial z} = 0. \quad (3)$$

The following dimensionless variables were introduced here: $\tau = \omega t$ is time, $yR = r$, $x = zL$ are the radial and longitudinal coordinates of the cylindrical coordinate system, $u = \bar{u}c_0$, $v = \bar{v}c_0$ are the longtitudinal and radial velocity components, $\rho = \rho_0(1 + \bar{\rho})$, $p = \rho_0 c_0^2(1 + \bar{p})$ are the density and the pressure, and the subscript 0 referes to unperturbed quantities. The system of equations (3) was obtained following expansion of the Navier–Stokes, continuity, and adiabat equations, written down in cylindrical coordinates, in powers of the small quantities $\bar{u}$ and ε. The effective viscosity coefficient ν_{eff} is the sum $\nu_{\text{eff}} = \nu + \nu_t$ (ν_t is the turbulent viscosity, for whose determination one uses the quasistationary hypothesis).

Integrating (3) over the cross section of the tube, and using the boundary conditions $\bar{v}(y = 1) = 0$, $\partial\bar{u}/\partial y = 0$ at $y = 0$, we obtain

$$\frac{R\omega}{c_0}\frac{\partial}{\partial \tau}\int_0^1 dy\,y\bar{u} = -\frac{\varepsilon}{2}\frac{\partial \bar{p}}{\partial z} + \frac{1}{c_0 R}\left[\nu_{\text{eff}}\, y\frac{\partial \bar{u}}{\partial y}\right]\Bigg|_{y=1},$$
$$\frac{1}{2}\frac{R\omega}{c_0}\frac{\partial \bar{p}}{\partial \tau} + \varepsilon\frac{\partial}{\partial z}\int_0^1 dy\,y\bar{u} = 0. \quad (4)$$

The solution of system (4) is sought in the form $\bar{u} = f(y)\varphi(z)\exp(i\tau)$, $\bar{p} = \psi(z)\exp(i\tau)$. Substituting it into (4), and assuming that ν_t depends on y only, we have

$$\bar{u}^{(1)} = if(y)Ce^{i\delta}\sin(k'z + \alpha)\exp(i\tau), \quad \bar{p}^{(1)} = 2a\sqrt{1 + i\beta}\,C\cos(k'z + \alpha)\exp(i\tau). \quad (5)$$

Here C, δ, and α are real and complex constants, $\beta = \gamma/a\omega R^2$; $k' = k\sqrt{1 + i\beta}$; $k = \omega L/c_0$; $a = \int_0^1 dy\,yf(y)$; $\gamma = [\nu_{\text{eff}}\, y\,df/dy]|_{y=1}$.

Let the oscillation source be located at $z = 0$, and let the coordinate of the other end of the tube be $z = 1$. The solutions (5) must satisfy the boundary conditions

$$\bar{u}_s(z = 0) = -iM_0\exp(i\tau) \quad (6)$$

($M_0 = u_0/c_0$ is the Mach number for the velocity oscillation amplitude of the source, and the subscript s denotes averaging over the tube cross section) and the boundary condition

$$\bar{u}_s(z = 1) = 0 \quad (7)$$

for a closed tube, or

$$\bar{p}_s(z = 1) = 0 \quad (8)$$

for a tube with open ends. The whole study is carried out for frequencies substantially lower than the resonance frequencies; therefore, condition (8) is fully acceptable. For the behavior of the more complicated nonlinear boundary condition see [9]. Substituting (5) into (6)-(8), and putting $\beta \ll 1$, one obtains for a closed tube

$$\alpha_0 = \operatorname{Real}\alpha = \pi - k, \quad \alpha' = \operatorname{Im}\alpha = -\frac{1}{2}k\beta, \quad \operatorname{tg}\delta = -\frac{k\beta}{2}\operatorname{ctg}k,$$
$$C = -\frac{M_0}{2a}\frac{\sqrt{1 + \operatorname{tg}^2\delta}}{\sin k - \frac{k\beta}{2}\operatorname{tg}\delta\cdot\cos k} \simeq -\frac{M_0}{2a}\frac{1}{\sin k}, \quad (9)$$

and for a tube open at one end

$$\alpha_0 = \frac{\pi}{2} - k, \quad \alpha' = -\frac{1}{2}k\beta, \quad \operatorname{tg}\delta = \frac{k\beta}{2}\operatorname{tg}k,$$
$$C = -\frac{M_0}{2a}\frac{\sqrt{1+\operatorname{tg}^2\delta}}{\cos k + \frac{k\beta}{2}\operatorname{tg}\delta\cdot\sin k} \simeq -\frac{M_0}{2a}\frac{1}{\cos k}. \quad (10)$$

It follows from (9) and (10) that the first resonance in a closed tube occurs at $k_0 \simeq \pi$, and in an open tube — at $k_0 \simeq \pi/2$.

We determine the profile f(y) in (5). It is assumed that near the tube wall (y = 1) there is a viscous sublayer in which $\nu_{eff} = \nu$, while at a distance $1 - y^*$ is found a region of evolving turbulence. It can then be assumed that in the turbulent flow core is transported the tangential stress τ_w existing at the wall, i.e., in the flow core

$$\tau_w = \tau_t \quad (11)$$

(τ_t is the tangential stress in the turbulent core). The assumption (11) is quite crude, but leads to good results in the case of stationary flows [6]. It is assumed that at each moment of time τ_w and τ_t are the same as for a stationary flow. This assumption is the essence of the quasistationarity hypothesis. Using the Prandtl equation [6] for τ_t, and the Blasius law [6] for τ_w, as well as averaging (11) over z and τ, we have

$$\langle\tau_w\rangle_\tau = \sigma^2 (R-r)^2 \rho \left\langle \left|\frac{\partial u}{\partial r}\right|^2 \right\rangle_{\tau z}, \quad \tau_w = \frac{\lambda\rho u_{me}^2}{8}, \quad \lambda = \frac{0{,}3164}{(2Ru_{me}/\nu)^{0{,}25}} \quad (12)$$

(u_{me} is the mean over cross section and length, and $\sigma = 0.4$ is an empirical constant). Substituting expression (5) into (12) for $\beta \ll 1$, and requiring that $f_1(y = 0) = 1$, the profile $f_1(y)$ is obtained in the turbulent core of the flow:

$$f_1(y) = 1 + E\ln(1-y), \quad E^2 = \frac{0{,}059}{\mathrm{Re}_c^{0{,}25}}\left(\frac{2al}{k}\right)^{1{,}75}\frac{\Gamma(0{,}375)}{\sigma^2 h\sqrt{\pi}}. \quad (13)$$

Here $\Gamma(x)$ is the Euler gamma-function, $\mathrm{Re}_c = 2RCc_0/\nu$; $h = 1 + \sin 2\alpha_0/2k$; $l = 2\sin(k/2 + \alpha_0)\sin(k/2)$. By means of (13) one can calculate a by neglecting the thickness of the viscous sublayer: $a = (1 - 3E/2)/2$.

The profile $f_2(y)$ in the viscous sublayer, where τ_w is constant in the stationary case, is

$$f_2(y) = \frac{1}{C}\frac{\langle\tau_w\rangle_\tau}{\mu c_0}(R-r) = D(1-y), \quad D = \frac{0{,}0074}{\sqrt{\pi}}\Gamma(0{,}375)\left(\frac{2al}{k}\right)^{1{,}75}\mathrm{Re}_c^{0{,}75} \quad (14)$$

It follows from (14) that $\beta = -\beta_0^2 D/2a$, $\beta_0 = (2\nu/\omega)^{1/2}/R$ in (5).

The radial velocity component is found from the equation of continuity. Imposing the boundary conditions v(y = 1) = 0, v(y = 0) = 0, we find

$$\overline{v}^{(1)} = -ig(y)\cos(k'z + \alpha)\exp(i\tau), \quad (15)$$

where the profiles $g_1(y)$ far from the wall and $g_2(y)$ near the wall are given by the expressions

$$g_1(y) = k'\varepsilon C\left\{y\left[a + \frac{1-E/2}{2}\exp(i\delta)\right] + \frac{E}{2}\left[\left(y - \frac{1}{y}\right)\ln(1-y) - 1\right]\exp(i\delta)\right\},$$
$$g_2(y) = k'\varepsilon C\left[a\left(y - \frac{1}{y}\right) + \frac{D}{2}\left(y - \frac{2}{3}y^2 - \frac{1}{3y}\right)\exp(i\delta)\right]. \quad (16)$$

The turbulent viscosity ν_t is determined as follows:

$$\nu_t = \sigma(R-r)\sqrt{\frac{1}{\rho}\langle\tau_w\rangle_\tau}, \quad \nu_{eff} = \nu\,[1 + w(1-y)],$$
$$w = \frac{1}{2}\sigma^2 C\,\mathrm{Re}_0 E\sqrt{h}, \quad \mathrm{Re}_0 = Rc_0/\nu. \quad (17)$$

In deriving Eqs. (17) we used the equation for the turbulent viscosity in a stationary flow and expressions (9), (10) for α_0.

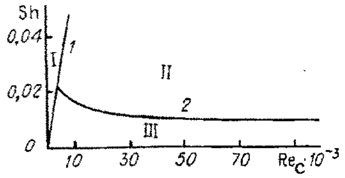

Fig. 1

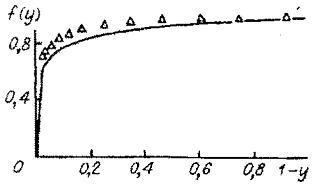

Fig. 2

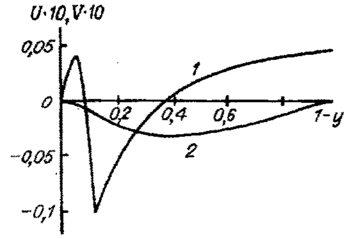

Fig. 3

In Fig. 2 we compare the theoretical profile obtained (13), (14) with experiment [10]. The dots in Fig. 2 correspond to the experimental value of the quantity $(u_{max} - u_{min})/u(y = 0)$, where u_{max} and u_{min} are the maximum and minimum velocity values for a given y during an oscillation period. This method of handling the experimental results makes it possible to eliminate the constant velocity component. It is seen from Fig. 2 that the theoretical profile is in satisfactory agreement with the experimental one. It is necessary to point out that expressions (13) and (14) were obtained under the assumption of quasistationary flow, i.e., for $Z < 1$, while in the experiments [10] $Z \sim 3$. The results of [5], however, make it possible to conclude that the effect of nonstationarity on the velocity profile becomes substantial only when $Z > 4$. The phase shift δ in Eqs. (5) is independent of y. The same results follows from the experimental data [10]. Thus, it can be stated that the expressions for $\bar{u}^{(1)}$ and f(y) are, on the whole, verified by experiment [10].

The secondary flows are described by the equations obtained within the second order of expanding the Navier–Stokes, continuity, and adiabat equations in power series of the small quantity $\bar{u}$, and are the solution of the system [11]

$$\frac{\partial \bar{p}^{(2)}}{\partial y} = 0, \quad \varepsilon \frac{\partial \bar{p}^{(2)}}{\partial z} - \frac{1}{RC_0} \frac{1}{y} \frac{\partial}{\partial y} \left(\nu_{\text{eff}} y \frac{\partial \bar{u}^{(2)}}{\partial y} \right) = -\varepsilon k \left\langle \bar{p}^{(1)} \frac{\partial \bar{u}^{(1)}}{\partial \tau} \right\rangle - \varepsilon \left\langle \bar{u}^{(1)} \frac{\partial \bar{u}^{(1)}}{\partial z} \right\rangle - \left\langle \bar{v}^{(1)} \frac{\partial \bar{u}^{(1)}}{\partial y} \right\rangle,$$
$$\frac{1}{y} \frac{\partial}{\partial y} (y \bar{v}^{(2)}) + \varepsilon \frac{\partial \bar{u}^{(2)}}{\partial z} = -\varepsilon \left\langle \frac{\partial}{\partial z} (\bar{p}^{(1)} \bar{u}^{(1)}) \right\rangle - \left\langle \bar{p}^{(1)} \frac{1}{y} \frac{\partial}{\partial y} (y \bar{v}^{(1)}) \right\rangle, \tag{18}$$

where the angular brackets denote averaging over time.

The solutions of Eqs. (18) must satisfy the following boundary conditions: on the tube wall at y = 1

$$\bar{u}^{(2)}(y = 1) = 0, \quad \bar{v}^{(2)}(y = 1) = 0; \tag{19}$$

and on the tube axis (y = 0) it is sufficient to require finiteness of $\partial \bar{u}^{(2)}/\partial y$ and $\bar{v}^{(2)}$ [6]. At $y = y^*$, where the profiles (13) and (14) are matched, one must put

$$\bar{u}_1^{(2)}(y = y^*) = \bar{u}_2^{(2)}(y = y^*), \quad \bar{v}_1^{(2)}(y = y^*) = \bar{v}_2^{(2)}(y = y^*) \tag{20}$$

($\bar{u}_1^{(2)}$ and $\bar{u}_2^{(2)}$ denote the solutions far from and near the wall, respectively). Finally, one more condition is imposed by the requirement that the fluid divergence in the secondary flow vanish:

$$\int_0^{y^*} \bar{u}_1^{(2)} y\,dy + \int_{y^*}^1 \bar{u}_2^{(2)} y\,dy = 0. \tag{21}$$

In the following calculations, to simplify the expressions we take into account that $\delta \sim \beta$, $E \ll 1$, and we neglect contributions $\sim\beta^2$, βE, $E \sin\delta$, and so on. Besides, as is seen from the numerical calculation, the contribution of terms proportional to $\sinh k\beta(z - 1)$, $\cosh k\beta(z - 1)$ is smaller than the contribution of the remaining terms by almost two orders of magnitude, and therefore these terms are omitted. For the same reason one may omit the terms proportional to $\beta/\text{Re}_0\varepsilon k$ ($\text{Re}_0\varepsilon k \gg 1$). Substituting expressions (5), (13)-(17) into (18) and invoking conditions (19)-(21), we obtain

$$\frac{\bar{u}_1^{(2)}}{\text{Re}_0 \varepsilon k C^2} = -\frac{1}{2w} \left\{ F_1(y) C_1(z) + \frac{E}{4} \left[2(1 - a\cos\delta) F_2(y) - 3\ln t - \right.\right.$$
$$\left.\left. - \frac{3}{w_1} F_3(y) \right] \sin 2(kz + \alpha_0) + C_2(z) \right\}, \quad t = 1 + w(1 - y), \quad w_1 = (w + 1)/w,$$

$$\frac{\bar{u}_2^{(2)}}{\text{Re}_0 \varepsilon k C^2} = \frac{C_1(z)}{4} y^2 + \frac{Dy}{8} [2a\cos\delta \cdot G_1(y) - DG_2(y)] \sin 2(kz + \alpha_0) +$$

$$+ C_2^*(z)\ln y + C_3^*(z), \quad G_1(y) = 1 + \frac{y}{2} - \frac{y^2}{3}, \quad G_2(y) = -\frac{1}{3} + \frac{y}{2} - \frac{y^2}{3} + \frac{y^3}{12},$$

$$\frac{\overline{v}_1^{(2)}}{\mathrm{Re}_0(\varepsilon k)^2 C^2} = \frac{1}{2wy}\left\{\frac{J_1(y)C_1'(z)}{k} + \frac{E}{2}\left[3J_4(y) + 2(1 - a\cos\delta)J_2(y) - \frac{3}{w_1}J_3(y)\right]\times\right.$$

$$\left.\times\cos 2(kz+\alpha_0) + y^2\frac{C_2'(z)}{2k}\right\} + \frac{C_3(z)}{y}, \quad \frac{\overline{v}_2^{(2)}}{\mathrm{Re}_0(\varepsilon k)^2 C^2} = -\frac{C_1'(z)y^3}{16k} -$$

$$-\frac{Dy^2}{4}\left[2a\cos\delta I_1(y) - DI_2(y)\right]\cos 2(kz+\alpha_0) - \frac{C_2^{*\prime}(z)}{2k}y\left(\ln y - \frac{1}{2}\right) + \frac{C_3^{*\prime}(z)}{2k}y + \frac{C_4^*(z)}{y}, \tag{22}$$

$$I_1(y) = \frac{1}{3} + \frac{y}{8} - \frac{y^2}{15}, \quad I_2(y) = -\frac{1}{9} + \frac{y}{8} - \frac{y^2}{15} + \frac{y^3}{72}.$$

The functions $F_i(y)$, $J_i(y)$, $C_i(z)$, $C_i^*(z)$ are given in the Appendix.

The result of the calculations performed can be written in the form

$$\frac{\overline{u}^{(2)}}{\mathrm{Re}_0\varepsilon k C^2} = U(y)\sin 2(kz+\alpha_0), \quad \frac{\overline{v}^{(2)}}{\mathrm{Re}_0(\varepsilon k)^2 C^2} = V(y)\cos 2(kz+\alpha_0).$$

Figure 3 provides plots of the functions U(y) and V(y) (curves 1 and 2, respectively). The calculation was carried out for a closed tube with the parameters Re_c = 70,000, Sh = 0.01, $\varepsilon = 0.5\cdot10^{-3}$, M_0 = 0.05, $k = \pi/4$ (with the resonance frequency being $k_0 = \pi$). As seen from Fig. 3, the secondary flow in a turbulent oscillatory flow is a ring vortex, filling the whole tube. In the region $0 \leqslant y \leqslant y^*$ the structure of this vortex recalls the structure of the Rayleigh vortex, formed in a laminar oscillatory flow [1]. Near the wall and for $y_* \leqslant y \leqslant 1$ the structure of this vortex recalls the structure of the Schlichting vortex in a laminar flow [6]. On the whole the qualitative flow pattern is similar to that in the case of laminar flow [11]. If, however, the velocity in the boundary layer vortex in a laminar flow does not exceed one third of the velocity at the center of the tube, then these velocities almost coincide in the case under consideration. This velocity matching is, most probably, related to the increase in the effective viscosity upon moving away from the tube wall.

It is well known that the pattern of secondary flows in a channel in the case of a laminar oscillatory flow can vary substantially if by transforming from Euler to Lagrange coordinates [3]; more precisely: the Schlichting vortex vanishes in Lagrange coordinates. The transition equations are

$$\overline{u}_L^{(2)} = \overline{u}^{(2)} + \frac{1}{k}\,\frac{\partial\overline{u}^{(1)}}{\partial z}\int\overline{u}^{(1)}d\tau + \frac{1}{k\varepsilon}\,\frac{\partial\overline{u}^{(1)}}{\partial y}\int\overline{v}^{(1)}d\tau\,,$$
$$\overline{v}_L^{(2)} = \overline{v}^{(2)} + \frac{1}{k}\,\frac{\partial\overline{v}^{(1)}}{\partial z}\int\overline{u}^{(1)}d\tau + \frac{1}{k\varepsilon}\,\frac{\partial\overline{v}^{(1)}}{\partial y}\int\overline{v}^{(1)}d\tau\,. \tag{23}$$

Substituting expressions (5) and (15) into (23), we obtain corrections $\Delta\bar{u}^{(2)}$ and $\Delta\bar{v}^{(2)}$ to (22). It turns out that their contribution is negligibly small far from the tube wall, therefore we write down their expressions in the boundary layer region:

$$\frac{\Delta\overline{u}^{(2)}}{\mathrm{Re}_0\varepsilon k C^2} = \frac{D}{8\,\mathrm{Re}_0\,\varepsilon k}\left[ad\left(\frac{1}{y} - y\right) - \beta D\left(\frac{1}{2} - 2y + \frac{1}{6y} + \frac{4}{3}y^2\right)\right]\sin 2(kz+\alpha_0),$$

$$\frac{\Delta\overline{v}^{(2)}}{\mathrm{Re}_0(\varepsilon k)^2C^2} = \frac{D}{8\,\mathrm{Re}_0\,\varepsilon k}\left\{(1-y)\left[ad\left(\frac{1}{y} - y\right) + \beta D\left(y - \frac{2}{3}y^2 - \frac{1}{3y}\right)\right] - \frac{2a}{3}\left(y^2 + \frac{2}{y} - 3\right)\sin\delta\right\}\cos 2(kz+a_0), \tag{24}$$

$$d = 2\sin\delta - \beta\cos\delta.$$

Different versions can be implemented in the case analyzed of turbulent flows due to a complicated dependence of the secondary flow velocity on the oscillation amplitude. Thus, the corrections (24) are small for the case illustrated in Fig. 3, and do not change the structure of the secondary flow.

In conclusion the author is grateful to R. G. Galiullin and V. B. Repin for their interest in this study and for their comments.

Appendix. The functions and constants appearing in Eqs. (22):

$$F_1(y) = y + w_1\ln(1 - y/w_1),$$

$$F_2(y) = (w_1 \ln t - 1 + y)\ln(1-y) - y + w_1 \mathrm{Li}_2(1-t),$$
$$F_3(y) = \ln(1-y)\ln(t/y) + \mathrm{Li}_2(1-t) - \mathrm{Li}_2(1-y),$$

$\mathrm{Li}_2(x)$ is the Euler dilogarithm; for $|x| < 1$ the dilogarithm is defined by the series

$$\mathrm{Li}_2(x) = \sum_{k=1}^{\infty} x^k/k^2,$$

for $|x| > 1$ and $|\arg(-x)| < \pi$ the analytic continuation is:

$$\mathrm{Li}_2(x) = \mathrm{Li}_2\left(\frac{1}{1-x}\right) + \frac{1}{2}\ln^2(1-x) - \ln(-x)\ln(1-x) - \frac{\pi^2}{6},$$

$$J_1(y) = \frac{1}{3}y^3 - \frac{w_1^3}{2}\left[\left(1-\frac{y^2}{w_1^2}\right)\ln\left(1-\frac{y}{w_1}\right) + \frac{y}{w_1}\left(1+\frac{y}{2w_1}\right)\right], \quad J_2(y) = -\frac{w_1}{4}(w_1^2 - y^2)\times$$
$$\times \ln^2 t + w_1\left[(w_1^2-y^2)\ln w + \frac{1}{4}\left(\frac{3w+2}{w} - 2\frac{w-1}{w}y - y^2\right)\right]\ln t + \frac{(1-y)^2}{6}(1+2y)\times$$
$$\times\ln(1-y) + \frac{1}{2}\left(w_1^2\ln w + \frac{w_1}{2}\frac{3w-1}{w} + \frac{1}{3}\right)y - \left(\frac{29}{36} - \frac{w_1}{4}\ln w - \frac{w_1}{8} + \frac{\pi^2 w_1}{16} + \frac{4}{9}y\right)y^2,$$

$$J_3(y) = -\frac{w_1^2-y^2}{4}\ln^2 t + \frac{1}{8}\left[(1-y)\left(\frac{5w+4}{w}+y\right) + 4(w_1^2-y^2)\ln w\right]\ln t + S(y) -$$
$$-\frac{1}{2}\mathrm{Li}_2(1-y) - \frac{1}{2}y^2\ln y\ln(1-y) - \frac{1-y^2}{4}\ln(1-y) + \frac{y}{4}(2+y)\ln y - \frac{3+2\pi^2}{24}y^2 +$$
$$+\frac{1}{2}\left(w_1\ln w - \frac{2w+1}{2w}\right)y, \quad J_4(y) = \frac{w_1^2-y^2}{2}\ln t + \frac{y}{4}(2w_1+y),$$

$$S(y) = \sum_{k=1}^{\infty}\frac{(1-y)^{k+1}}{k^2}\,\frac{1+(k+1)y}{(k+1)(k+2)},$$

$C_1(z) = \partial\bar{p}^{(2)}/\partial z$, and in calculating $C_1(z)$ the following boundary condition $\partial\bar{p}^{(2)}/\partial z = 0$ is imposed at $z = 1$

$$C_1'(z) = \frac{4k}{1-y^{*2}}\left[-y^*\Phi_5(z) + C_3(z) - \frac{1}{k}\Phi_1'(z) + \Phi_4(z)\right] - 4\Phi_2'(z), \quad C_2(z) =$$
$$= \frac{4w}{1-y^{*2}}\left\{-\mu_2 C_1(z) + 2\ln y^*\cdot\Phi_1(z) + \mu_3\Phi_3(z) - \left[\frac{1}{2} + y^{*2}\left(\ln y^* - \frac{1}{2}\right)\right]\Phi_3(z)\right\},$$

$$C_3(z) = -\frac{E}{4w}\left[3A_4 + 2A_2(1 - a\cos\delta) - \frac{3}{w_1}A_3\right]\cos 2(kz+\alpha_0),$$

$$C_2^*(z) = \frac{2}{1-y^{*2}}\left[\mu_1 C_1(z) - 2\Phi_1(z) - \Phi_2(z) + y^{*2}\Phi_3(z)\right],$$

$$C_3^*(z) = -\frac{1}{4}C_1(z) - \Phi_2(z), \quad C_4^*(z) = \left(\frac{3}{8} - \frac{\mu_1}{1-y^{*2}}\right)\times$$
$$\times\frac{C_1'(z)}{2k} - \Phi_4(z) + \frac{1}{2k(1-y^{*2})}\left[2\Phi_1'(z) + (2-y^{*2})\Phi_2'(z) - y^{*2}\Phi_3'(z)\right],$$

$$A_2 = \frac{3w_1}{4}(\ln w + 1), \quad A_3 = \frac{1}{4}\left(w_1^2\ln w + \frac{5w+2}{2w}\right)\ln(1+w) + S(0) - \frac{\pi^2}{12}, \quad A_4 =$$
$$= \frac{w_1^2}{2}\ln(1+w), \quad \mu_1 = -\frac{1-y^{*4}}{8} + \frac{1}{w}\left[\frac{y^{*2}}{2}F_1(y^*) - J_1(y^*)\right], \quad \mu_2 = -\frac{1-y^{*4}}{8}\ln y^* -$$
$$-\frac{(1-y^{*2})^2}{8} - \frac{J_1(y^*)}{w}\ln y^* + \frac{F_1(y^*)}{4w}\left[1 + 2y^{*2}\left(\ln y^* - \frac{1}{2}\right)\right], \quad \mu_3 = \frac{1-y^{*2}+2\ln y^*}{2},$$

$$\Phi_1(z) = \frac{E}{8w}\left\{2(1-a\cos\delta)\left[J_2(y^*) - A_2\right] - 3\left[J_4(y^*) - A_4\right] - \frac{3}{w_1}\left[J_3(y^*) - A_3\right]\right\}\times$$
$$\times\sin 2(kz+\alpha_0) - \frac{D}{8}\left\{2a\cos\delta\left[\frac{47}{120} - y^{*3}I_1(y^*)\right] + D\left[\frac{7}{180} + y^{*3}I_2(y^*)\right]\right\}\sin 2(kz+\alpha_0),$$

$$\Phi_2(z) = \frac{D}{48}\left(14a\cos\delta + \frac{D}{2}\right)\sin 2(kz + \alpha_0),$$

$$\Phi_3(z) = \frac{E}{8w}\left[2(1 - a\cos\delta)F_2(y^*) - 3\ln t^* - \frac{3}{w_1}F_3(y^*)\right]\sin 2(kz + \alpha_0) +$$

$$+ \frac{D}{8}y^*\left[2a\cos\delta\cdot G_1(y^*) - DG_2(y^*)\right]\sin 2(kz + \alpha_0),$$

$$\Phi_4(z) = -\frac{D}{240}\left(47a\cos\delta + \frac{7D}{3}\right)\cos 2(kz + \alpha_0),$$

$$\Phi_5(z) = -\frac{E}{4wy^*}\left[3J_4(y^*) + 2(1 - a\cos\delta)J_2(y^*) - \frac{3}{w_1}J_3(y^*)\right]\cos 2(kz + \alpha_0) -$$

$$-\frac{Dy^{*2}}{4}\left[2a\cos\delta\cdot I_1(y^*) - DI_2(y^*)\right]\cos 2(kz + \alpha_0).$$

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RADIATION OF INTERNAL GRAVITATIONAL WAVES IN THE CASE OF UNIFORM MOTION OF SOURCES OF VARIABLE AMPLITUDE (THE PLANE PROBLEM)

V. A. Gorodtsov

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A uniformly moving source generates waves similar to Cherenkov radiation. In a liquid with stratified density these are internal gravitational waves. A fixed oscillating source generates another type of radiation of gravitational waves. When a source of variable amplitude is moving the variety of excited waves increases. Wave-forerunners appear which carry away energy in the direction of motion with a velocity exceeding the velocity of the source.

The linear wave fields around an oscillating moving source were analyzed in [1-7] for the simplest types of stratification, a free surface and a discontinuous jump in the density. Below we estimate the energy losses of such sources for a more general form of stratification. The method of energy estimates also enables one to investigate more simply the main known and certain additional features of the radiation in the case of discontinuous stratification.

In considering a mass source with an harmonically varying amplitude, moving uniformly horizontally in a stratified incompressible liquid, we will confine ourselves to analyzing the

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